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RESEARCHING TEACHERS'KNOWLEDGE FOR TEACHING MATHEMATICS[1]

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Resumo
Pesquisando o conhecimento docente para o ensino da Matemática
Este relatório teoriza e fornece evidência empírica de como pesquisadores e educadores podem reconhecer as categorias de conhecimentos dos professores para o ensino, como professores ensinam e discutem com seus pares o comportamento matemático do aluno e sua prática. Sua orientação teórica envolve os trabalhos de Shulman sobre o conhecimento do conteúdo pedagógico, de Ball and Bass sobre o conhecimento matemático para o ensino, e de Steinbring sobre os conhecimentos epistemológicos dos professores. A evidência empírica surge a partir da prática dos professores que trabalham com a classe trabalhadora de Latinos e Afro-Americanos e estudantes em uma escola de bairro pobre urbano nos Estados Unidos da América. Os resultados desta investigação, parte de uma investigação mais ampla, sugerem que as categorias de conhecimentos dos professores se implicam mutuamente.

Palavras-chave: ensino aprendizagem, formação do professor, prática docente, ensino da matemática

Abstract
This report theorizes and provides empirical evidence of how researchers and educators might recognize categories of teachers' knowledge for teaching as teachers teach and discuss with peers their student's mathematical behavior and their practice. Its theoretical orientation engages work by Shulman on pedagogical content knowledge, Ball and Bass on mathematical knowledge for teaching, and Steinbring on teachers' epistemological knowledge. The empirical evidence emerges from the practice of teachers working with working class African American and Latino students in a poor, urban school district in the United States of America. The results of this investigation, part of larger,

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broader inquiry, suggest that the categories of teachers' knowledge implicate each other.

Key words: teaching and learning, teacher training, teaching practice, teaching of mathematics

Introduction

Teaching mathematics is a multifaceted human endeavor, involving a complex, moment-by-moment interplay of different categories of knowledge. Teachers' mathematical knowledge, pedagogical competence, and insight into the development of students' mathematical ideas and reasoning are key to improving students' mathematical achievement. High quality standards, curriculum, instructional materials, and assessments are also important but not enough to improve students' learning of mathematics. As Ball, Hill and Bass (2005) argue, "little improvement is possible without direct attention to the practice of teaching … [h]ow well teachers know mathematics is central" (p. 14). Conceivably, this explains why recently there has been considerable discussion and research on teachers' subject-matter knowledge, pedagogical content knowledge, and mathematical knowledge for teaching (for example, Adler & Davis, 2006; Ball, 2000; Fennema et al., 1996; Hill, Rowan, & Ball, 2005; Shulman, 1986). The problem we theorize and explore empirically is "How might educators and researchers investigate and understand the development of teachers' mathematical knowledge for teaching?" Our perspective seeks descriptions of how teachers develop their mathematics knowledge for teaching in the complex, discursive interaction of actual practice as students evidence their mathematical ideas and reasoning and in the course of teachers' discussion of students' mathematical behaviors.

Theoretical Perspective

The theoretical perspective for our methodological approach has several sources. It is based on the assumption that teachers engage several categories of knowledge to enact successfully the mathematics education of their students. They clearly need knowledge of mathematics as well as knowledge of the subject that is specific to their work as teachers. In agreement with Shulman (1986) and Ball, Hill, and Bass (2005), our perspective recognizes that to teach a school subject like mathematics effectively necessitates knowledge of mathematics that "goes beyond the knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p. 9), or what Ball (2000) terms "mathematical knowledge for teaching. In their practice, teachers also need management and organizational knowledge that is distinct from subject matter knowledge—pedagogical knowledge (Shulman, 1987). Furthermore, effective teaching requires teachers to attend to and endeavor to understand the mathematical ideas and reasoning of their students (Maher, 1998; Sowder, in press). This category of knowledge is specific and varies moment-to-moment and refers to teachers' inference into the

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status of students' knowledge. As Steinbring (1998) notes, a teacher "has to become aware of the specific epistemological status of the students' mathematical knowledge. … to diagnose and analyze students' constructions of mathematical knowledge and … to compare those constructions to what was intended to be learned in order to vary the learning offers accordingly" (p. 159). This category of knowledge—teachers' awareness of the epistemological status of students' mathematical understanding—enables, for instance, teachers to pose appropriate, new challenges for students to consider as they further build mathematical ideas and reasoning.

Researchers can infer teachers' mathematical knowledge for teaching by analyzing their practice in action, including interactions with students, questions they ask, issues they make salient to students, student artifacts they use, as well as post-session analyses they perform of their actions, plans, and students' work. Interaction also provides a lens through which to view mathematical knowledge, mathematical knowledge for teaching, pedagogical knowledge, and awareness of the epistemological status of students' mathematical understanding. These four categories of knowledge though conceptually different do at times, as we have observed, interact and even intersect. When they do intersect, they are essentially indistinguishable one from the other. Teachers' mathematical knowledge for teaching can be observed through their pedagogical moves; that is, by way of their pedagogical knowledge revealed in their moment-to-moment discursive interaction with students. In this paper, based on our methodological approach, we provide empirical evidence to substantiate the theoretical claim that teachers' mathematical knowledge, mathematical knowledge for teaching, pedagogical knowledge, and awareness of the epistemological status of students' mathematical understanding are in some instances mutually constitutive.

Method

This study is an adjunct of larger, ongoing analyses that emerge from a multi-prong, three-year research endeavor, "Informal Mathematics Learning Project" (IML). Two primary goals of the IML project involve investigating (1) how middle-school students (11 to 13 years old) develop mathematical ideas and reasoning over time in an informal, after-school environment and exploring relationships between agency and students' learning as well as (2) how teachers facilitate IML sessions and attend to students' ideas and reasoning. The IML research sessions occur in a middle school, after-school program in Plainfield, New Jersey, an economically depressed, urban area, whose school population is 98 percent African American and Latino students. These sessions were held after the regular school day to avoid some constraints of schools, such as time, curriculum, and testing.

For an academic year and a half, including the intervening summer, three pairs of teachers facilitated 20 sessions, 90-minute each, with a cohort of approximately 20 students, who began in their sixth grade, while graduate students from Rutgers University observed as ethnographers. This cohort explored similar mathematical tasks that had engaged an earlier cohort

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of students with whom researchers from Rutgers University worked, while the teachers participated as observers, taking field notes, and as co-investigators in post-session debriefings. Nonetheless, the teachers were not given a script; rather, they developed their own by selecting tasks and planning their own sessions. For about 50 minutes after each research session, the two teachers who facilitated the session, the other four teachers who observed the session, and two to three graduate students along with one Rutgers researcher discussed their observations and reflections on the tasks and on the ideas and reasoning of students. Research and debriefing sessions were videotaped. Students' inscriptions, graduate students' observation notes, teachers' planning documents were collected and stored electronically.

Through the course of IML sessions, the teachers invited students to work on strands of mathematical tasks. These tasks range across areas of mathematics that include rational numbers, combinatorics, probability and data analysis, and algebra. By design, the tasks are open-ended and well-defined, in that students were invited to determine what to investigate and how to proceed, identify patterns and search for relationships, make and investigate mathematical conjectures, develop mathematical arguments to convince themselves and others of their solutions, and evaluate their own arguments and those of others.

To understand the nature and development of mathematical knowledge for teaching, we analyzed data from the teachers' planning, implementation, debriefing sessions, as well as teachers' written reflections on the sessions they facilitated. For this report, we present an analysis of the first two IML sessions that two teachers, Lou (six years teaching experience) and Gilberto (three years teaching experience), facilitated as well as the corresponding work of students. For each session, there were between three and five video cameras, each with a boom microphone, capturing images from different student work groups and whole class discussions. Our videodata analysis follows methodological suggestions outlined by Powell, Francisco, and Maher (2003) and within this framework, we coded all data inductively and deductively. Our initial coding scheme intended to flag instances of teachers' using, commenting, and questioning about mathematics and pedagogy. Analyzing the data to understand teachers' mathematical knowledge for teaching, we noticed several instances of an intersection among teachers' awareness of the epistemological status of students' mathematical understanding and teachers' pedagogical and mathematical knowledge , some of which we present in the following section.

Results

The purpose of this paper is to theorize and explore an emergent approach for understanding the nature and development of teachers' knowledge. Above, we described a method for flagging critical events from data that provide investigators with insight on teachers' content and pedagogical knowledge as well as their awareness of the epistemological status of students' mathematical understanding. This section describes how we applied our

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methodology. Space only permits us to present a sequence of four critical events, occurring in one debriefing session.

The sequence of critical events concern students' presentation of ideas and teachers grappling with how understand the students' ideas and the underlining reasoning and how to orchestrate the next session based on the students' discourse that transpired in that day's after school session. These critical events provide us a window into the teachers' knowledge of pedagogy, mathematics, mathematics for teaching, and epistemological status of student learning. In the research session, students worked on the following task with Cuisenaire rods: If the light green rod has the number name two, what is the number name for the dark green rod? Three individual students each presented a different solution at an over head projector.

Tiffany stated that since the light green rod has the number name two, then the white, the red, the purple, the yellow, and the dark green rods have respectively the number names zero, one, three, four, and five. Devon asserted that the white, the red, the light green, the purple, the yellow, and the dark green rods have the number names one, two, three, four, five, and six, respectively. With different results, both Tiffany and Devon lined-up their rods according to their heights and used their ordinal position to reason what number names to assign the rods. The third student, Sameerah, reasoned that since light green has the number name two, then the dark green has the number name four because two light green rods have the same length of one dark green rod and therefore, two plus two is four. Her reasoning is based on the additive property of length (two light green rods placed end-to-end are equivalent in length to a dark green rod) to name the dark green rod four. The session concluded with Lou and Gilberto asking the students to think about the three different solutions and announcing that the following day they will revisit them.

The first critical event occurs during the debriefing session when Alice, a university researcher, asks the teachers to assess the validity of the three student solutions described above.

Alice: What do we think? Are they all equally valid?

Teacher1: Yes

Teacher2: No

Alice: Okay, I'm hearing, some -[4] Jennifer what do you think?

Jennifer: I guess I would respond to your question by saying "yes". They were valid to me because they were able to explain and justify their thinking behind it. It's not necessarily how I would have interpreted…i But that - the way the students who just lined them up in steps and explained to me, that if you call

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[4] In the quoted transcripts, the symbol "-" indicates pause in speech, "…" indicates inaudible speech, and bracketed words provide background information.

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this one two and the ones below it are one step down each and the ones above it - to me that's a reasonable explanation and justification and explanation for their reasoning.

Alice: Okay for their reasoning, now help me to understand what's going on in their reasoning

Jennifer: Using just the attribute of length. That's all they were looking at and it made sense to me. If this one is a certain number

Alice: Ok, if – you just said attribute of length

Jennifer: Yes, those weren't their words

Alice: Wait just a minute, though. What would the attribute of length be if you gave something the number name two?

The above discussion was selected as a critical event because the teachers begin to assess the validity of the students' solutions. By so doing, they discuss the students' reasoning and try to make sense of the students' understanding. They are exhibiting their mathematical knowledge for teaching and discussing their awareness of the epistemological status of students' mathematical understanding.

When Jennifer says that the students are using "just the attribute of length" to solve the task, Alice asks the teachers to discuss the mathematical meaning of assigning the number name two based on the attribute of length. The above discussion continues and is flagged as a second critical event:

Jennifer: Because the number name two –

Alice: What does that mean in relation to the attribute of length?

Kim: It depends on the unit.

Jennifer: It means that if I have one [rod] that's shorter, it's a number that going to be less than two and if it's longer then it's going to be greater than two.

Alice: Oh, okay, well that is certainly one thing that it means. Uhm, let's really push on this because mathematically, this is what we're having to agree on. You've all have said, and agreed, that it is the attribute of length that we are interested in and according to the attribute of length, you're giving light green the number name two. What does that mean about - anything? To say that it is two in length

Gilberto: That it is one plus one

Alice: It is one what… yeah I agree, but one what?

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Gilberto: One unit

Danielle: What ever the length It is

Kim: Unit

Alice: Okay, if this is two, then it is one unit plus one unit in length.

Here, a shift occurs in the conversation. Alice asks the teachers to consider what it means mathematically to give a rod the number name two. Hence, the discussion shifts from assessing the validity of the students' solutions and understanding their reasoning to a discussion of the underlying mathematical ideas of the task. While we hear a couple of the teachers use the term "unit" to answer Alice's question, we also hear responses that involve comparing lengths and applying the additive property. As a community, the teachers discuss and negotiate their mathematical knowledge as it pertains to comparing the lengths of Cuisenaire rods.

Once the teachers agree on the mathematical consequence of giving the number name two to the light green rod, they return to the assessing validity of the students' solutions. This is a shift in conversation from mathematics back to assessing the validity of students' reasoning is our third critical event.

Danielle: So based on that [Tiffany's] argument, the young lady that assigned, that said we're going to take the white off and were going to have that be zero then we're going to make red one and, and light green two and what was next, yellow three, and purple or purple then yellow and then she named it five, if we're looking at length, then its not, you can't justify that based on that answer because, you know, if light green is two then red should be half of light green, if, right, as one, if light green is two then yellow should have been exact, a double, she was naming yellow four, so yellow can't be four, however, because its not the double measure –

Jennifer: So it's not the attribute of length, it's the attribute of position.

In this discussion, Danielle notes the contradictions that would occur if yellow is given the number name four and red the number name one. In the first critical event, Jennifer assessed that the students were using the attribute of length to justify their solutions. After the discussions that occurred during the second critical event, Jennifer returns to correct her first statement and assert that the students are using the attribute of position to justify their solutions. This evidences the use of teachers' knowledge of mathematics to understand the status of the students' mathematical reasoning.

With this awareness of the epistemological status of students' knowledge and reasoning, Gilberto turns to designing an intervention for the next after school session, which we have flagged as our fourth critical event:

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Gilberto: [He moves to the overhead projector and lines Cuisenaire rods in height order: red, light green, purple, yellow, dark green.] So what they are taking into account here is order - the position, first, second, third, fourth, and then this [pointing to dark green rod] is the fifth position. So then, I say, well if it [light green rod] is two, then the number name for red will be one. She is going to, she might answer that. Then I will ask her, if this is one [pointing to red] - ok if this is two [pointing to light green], then she is going to say this is one [pointing to red], and then I think we should ask her, how many ones do you need to make two? And then she will probably come with something like this [aligning two red rods end-to-end and placing them adjacent to one light green rod] and then we will see that if red is called one, two ones will be bigger than two.

Prior to this statement, Gilberto acknowledges that the students must take the lengths of the rods into account to successfully progress through the Cuisenaire tasks that the teachers have planned over the following five after school sessions. He uses Tiffany's solution as a starting point for designing an intervention that he hopes will lead the students to recognize a contradiction and shift their reasoning from positional to additive. Gilberto's understanding of the underlying mathematical ideas of the task, his pedagogical intervention for the upcoming sessions, and his insights into the students' reasoning evidences specific aspects of his knowledge of pedagogy, mathematics, mathematics for teaching, and awareness of the epistemological status of students' mathematical understanding.

Discussion

By analyzing teachers' practices in action, we notice that their knowledge of mathematics, mathematics for teaching, and pedagogy, as well as their awareness of student learning intertwine and intersect. As the teachers discussed the students' reasoning and tried to make sense of the students' understanding, they exhibited their mathematical knowledge for teaching and their epistemological awareness of the students' mathematical understandings. In these conversations, their discussion shifted from assessing the validity of the students' solutions to an examination of the underlying mathematical ideas of the task. Consequently, as a community, the teachers negotiated their mathematical understanding, which they applied as they returned to assessing the validity of the students' solutions and discussing the epistemological status of students' knowledge. Using the shared knowledge of the group, Gilberto designed an intervention comprised of pedagogical moves informed by the teachers' collective understanding of possible student trajectories. These pedagogical moves included his awareness of the epistemological status of students' knowledge. Furthermore, his pedagogical intervention was based on developing a proof by contradiction (if the red rod has the number name 1, then since the length of two red rods, whose combined length is 2, is longer than the length of one light green rod, whose length is also 2) evidences his mathematical knowledge. Cumulatively, we are able to infer his mathematical knowledge for teaching from his pedagogical moves. His four knowledge

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domains (mathematics, mathematics for teaching, pedagogy, and awareness of the epistemological status of students' mathematical understanding) interact, one influencing the other.

We have found that researchers can acquire an understanding of four types of knowledge of teachers—teachers' knowledge of pedagogy, mathematics, mathematics for teaching, and epistemological awareness of students' mathematical—understanding by studying teachers' practice and their reflections on their practice. Specifically, researchers can obtain insights into the development of teachers' knowledge by observing how teachers analyze students' mathematical behavior, grapple with the mathematical, epistemological, and pedagogical issues involved in addressing challenges they perceive in facilitating students' growth in students' mathematical ideas and reasoning, as well as by studying teachers' pedagogical moves (Powell & Hanna, 2006).

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